

A new kind of solutions of the Einstein's field equations with non-vanishing cosmological constant

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Abstract

In this paper we construct a new kind of solutions, in particular the space-time-periodic solutions of the Einstein's field equations with non-vanishing cosmological constant, which possess some interesting physical properties. The singularities of these solutions are investigated and some new physical phenomena are discovered.

Key words and phrases: Einstein's field equations, cosmological constant, space-time-periodic solution, Riemann curvature tensor, Weyl scalars, singularity.

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1 Introduction

The Einstein's field equations with cosmological constant take the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (1.1)$$

where $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) is the unknown Lorentzian metric, $R_{\mu\nu}$ is the Ricci curvature tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature in which $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, Λ is the cosmological constant, G stands for the Newton's gravitational constant, c is the velocity of the light and $T_{\mu\nu}$ is the energy-momentum tensor. In a vacuum, i.e., in regions of space-time in which $T_{\mu\nu} = 0$, the Einstein's field equations (1.1) reduce to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0, \quad (1.2)$$

or equivalently,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \quad (1.3)$$

In cosmology, the cosmological constant Λ was proposed by Einstein as a modification of his original theory of general relativity to achieve a stationary universe. Unfortunately, Einstein abandoned the concept after the observation of the Hubble redshift indicated that the universe might not be stationary, since he had based his theory off the idea that the universe is unchanging (see [5]). However, the discovery of cosmic acceleration in the 1990s has renewed the interest in a cosmological constant. In this paper, we will construct a new kind of solutions, in particular the space-periodic solutions of the Einstein's field equations (1.2) (or equivalently (1.3)) with non-vanishing cosmological constant.

Recently, Kong and Liu [2] presented an effective new method to find the exact solutions of the Einstein's field equations, and by using this method they constructed some new exact solutions including the time-periodic solutions. The time-periodic solutions presented in [2] can be used to describe a regular space-time, which has vanishing Riemann curvature tensor but is inhomogenous, anisotropic and not asymptotically flat. In the series work [3], the authors constructed several kinds of new time-periodic solutions of the vacuum Einstein's field equations whose Riemann curvature tensors vanish, keep finite or take the infinity at some points in these space-times, respectively. However the norm of Riemann curvature tensors of all these solutions vanishes. This implies that these solutions essentially describe regular time-periodic space-times. More recently, Kong, Liu and Shen [4] constructed a new time-periodic solution of the vacuum Einstein's field equations. For the solution described in [4], not only its Riemann curvature tensor takes the

infinity at some points, but also the norm of the Riemann curvature tensor also go to the infinity at these points. Therefore, this solution possesses some physical singularities. Moreover, it has been shown that this solution is intrinsically time-periodic and then can be used to describe a time-periodic universe with the “time-periodic physical singularity”. By calculating the Weyl scalars of this new solution, some new physical phenomena are discovered and new singularities for the universal model described by our new solution has been analyzed.

In this paper we investigate the vacuum Einstein’s field equations with non-vanishing cosmological constant, namely, the equations (1.2) or (1.3). We first construct a new kind of solutions to (1.2), in particular including the space-time-periodic solution, then analyze the properties of curvature tensors, e.g., Riemann curvature tensor, Weyl scalars and so on, finally investigate the singularities of these solutions. Some new physical phenomena are also discovered.

The paper is organized as follows. In Section 2 we present our method to construct the new kind of solutions to the vacuum Einstein’s field equations with non-vanishing cosmological constant. Section 3 is devoted to finding the space-periodic solutions. In Section 4 we construct the space-time-periodic solutions. Some interesting properties enjoyed by our new solutions are analyzed and some new physical phenomena are discovered and investigated. A summary and some discussions are given in Section 5.

2 New solutions

In this section, we first construct a new kind of solutions to the equations (1.3), then calculate the Riemann curvature tensors, the norm of the Riemann curvature tensors and then analyze the Weyl scalars of the solutions constructed.

In the coordinates (t, x, y, z) , we consider the solution of the equations (1.3) with the following form

$$(\eta_{\mu\nu}) \triangleq \begin{pmatrix} b & a_x & 0 & 0 \\ a_x & 0 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix}, \quad (2.1)$$

where $a = a(t, x)$ and $b = b(t, x)$ are two smooth functions of t and x . (2.1) is a special case of Type I mentioned in Kong and Liu [2].

A direct calculation gives

$$R_{22} = R_{33} = \frac{2aa_{xt} + 2a_xa_t - b_xa - a_xb}{a_x}. \quad (2.2)$$

Solving the equation $R_{22} = \Lambda g_{22}$ or $R_{33} = \Lambda g_{33}$ yields

$$b = 2a_t + \frac{\Lambda a^2}{3} + \frac{f}{a}, \quad (2.3)$$

where $f = f(t)$ is an integral function. Substituting (2.3) into $R_{00} = \Lambda g_{00}$ leads to

$$\frac{f_t}{a^2} = 0, \quad (2.4)$$

that is to say,

$$f = m, \quad (2.5)$$

where m is a positive integral constant. Then the equation (2.3) becomes

$$b = 2a_t + \frac{\Lambda a^2}{3} + \frac{m}{a}. \quad (2.6)$$

It is easy to verify that if b is given by (2.6), then the Lorentzian metric¹ (2.1) solves

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \quad (\mu, \nu = 0, 1, 2, 3). \quad (2.7)$$

Thus, by the above argument, we obtain

Theorem 2.1 *The vacuum Einstein's field equations (1.2) have the following solution in the coordinates (t, x, y, z)*

$$ds^2 = \left(2a_t + \frac{\Lambda a^2}{3} + \frac{m}{a}\right) dt^2 + 2a_x dt dx - a^2(dy^2 + dz^2). \quad (2.8)$$

where a is an arbitrary function of t and x .

When $\Lambda = 0$, the metric (2.8) becomes

$$ds^2 = \left(2a_t + \frac{m}{a}\right) dt^2 + 2a_x dt dx - a^2(dy^2 + dz^2), \quad (2.9)$$

which is the solution of the vacuum Einstein's field equations

$$R_{\mu\nu} = 0.$$

By direct calculations, the scalar curvature R of the metric (2.8) equals to 4Λ , and the Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ reads

¹See [2].

$$R_{0101} = -\frac{a_x^2(\Lambda a^3 + 3m)}{3a^3}, \quad (2.10)$$

$$R_{0202} = -\frac{12\Lambda a^4 a_t - 18ma a_t + 2\Lambda^2 a^6 + 3\Lambda m a^3 - 9m^2}{18a^2}, \quad (2.11)$$

$$R_{0212} = -\frac{a_x(2\Lambda a^3 - 3m)}{6a}, \quad (2.12)$$

$$R_{0303} = -\frac{12\Lambda a^4 a_t - 18ma a_t + 2\Lambda^2 a^6 + 3\Lambda m a^3 - 9m^2}{18a^2}, \quad (2.13)$$

$$R_{0313} = -\frac{a_x(2\Lambda a^3 - 3m)}{6a}, \quad (2.14)$$

$$R_{2323} = \frac{\Lambda a^4}{3} + am \quad (2.15)$$

and the other $R_{\alpha\beta\mu\nu} = 0$. Moreover, the norm of the Riemann curvature tensor is

$$\mathbf{R} \triangleq R^{ijkl} R_{ijkl} = \frac{8\Lambda^2 a^6 + 36m^2}{3a^6} \quad (2.16)$$

In general relativity, the Weyl scalars are a set of five complex scalar quantities Ψ_0, \dots, Ψ_4 describing the curvature of a four-dimensional space-time. They are the expressions of the ten independent degrees of freedom of the Weyl tensor $C_{\mu\nu\alpha\beta}$ in the Newman-Penrose formalism. According to the physical interpretation in Szekeres [7], Ψ_0 and Ψ_3 are ingoing and outgoing longitudinal radiation terms while Ψ_1 and Ψ_4 are ingoing and outgoing transverse radiation terms, and Ψ_2 is a *Coulomb term* representing the gravitational monopole of the source. By GRTENSOR II program, the Weyl scalars of (2.8) read

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \quad \Psi_2 = -\frac{2\Lambda a^3 - 3m}{18a^3}. \quad (2.17)$$

Remark 2.1 *Substituting $\Lambda = 0$ into (2.10)-(2.17), we obtain the Riemann curvature tensor, the norm of the Riemann curvature tensors and the Weyl scalars of the solution metric (2.9) to the vacuum Einstein equations with vanishing cosmological constant.*

3 Space-periodic solutions

This section is devoted to investigating the properties of the solution metric (2.8) and construct the space-periodic solutions to the equations (1.3).

Case 1: $m = 0$

In the present situation, the metric (2.8) becomes

$$ds^2 = \left(2a_t + \frac{\Lambda a^2}{3}\right) dt^2 + 2a_x dt dx - a^2(dy^2 + dz^2). \quad (3.1)$$

Define

$$\eta = t, \quad \xi = \frac{\Lambda t}{3} - \frac{2}{a}. \quad (3.2)$$

If

$$a_x \neq 0, \quad (3.3)$$

then in the coordinates (η, ξ, y, z) , the metric (3.1) takes the following form

$$ds^2 = a^2(d\eta d\xi - dy^2 - dz^2). \quad (3.4)$$

Introduce

$$\tilde{t} = \frac{\xi + \eta}{2}, \quad \tilde{x} = \frac{\xi - \eta}{2}. \quad (3.5)$$

then in the coordinates $(\tilde{t}, \tilde{x}, y, z)$, the metric (3.4) becomes

$$ds^2 = a^2(d\tilde{t}^2 - d\tilde{x}^2 - dy^2 - dz^2), \quad (3.6)$$

where

$$a = \frac{6}{\tilde{t}(\Lambda - 3) - \tilde{x}(\Lambda + 3)}.$$

(3.6) implies that the metric (3.1) is conformal to the Minkowski space-time.

Case 2: $m \neq 0$

In particular, taking

$$a = 2 + \cos x. \quad (3.7)$$

we observe that the metric (2.8) becomes

$$ds^2 = \left(\frac{\Lambda(2 + \cos x)^2}{3} + \frac{m}{2 + \cos x}\right) dt^2 - 2 \sin x dt dx - (2 + \cos x)^2(dy^2 + dz^2). \quad (3.8)$$

It is easy to see that (3.8) is a Lorentzian metric (see Kong and Liu [2]) and describes a space-periodic space-time.

Substituting (3.7) into (2.10)-(2.15) gives

$$R_{0101} = -\frac{(\sin x)^2[\Lambda(2 + \cos x)^3 + 3m]}{3(2 + \cos x)^3}, \quad (3.9)$$

$$R_{0202} = -\frac{2\Lambda^2(2 + \cos x)^6 + 3\Lambda m(2 + \cos x)^3 - 9m^2}{18(2 + \cos x)^2}, \quad (3.10)$$

$$R_{0212} = \frac{\sin x [2\Lambda(2 + \cos x)^3 - 3m]}{6(2 + \cos x)}, \quad (3.11)$$

$$R_{0303} = -\frac{2\Lambda^2(2 + \cos x)^6 + 3\Lambda m(2 + \cos x)^3 - 9m^2}{18(2 + \cos x)^2}, \quad (3.12)$$

$$R_{0313} = \frac{\sin x [2\Lambda(2 + \cos x)^3 - 3m]}{6(2 + \cos x)}, \quad (3.13)$$

$$R_{2323} = \frac{\Lambda(2 + \cos x)^4}{3} + (2 + \cos x)m \quad (3.14)$$

and the other $R_{\alpha\beta\mu\nu} = 0$. Substituting (3.7) and into (2.16) leads to

$$\mathbf{R} = \frac{8\Lambda^2(2 + \cos x)^6 + 36m^2}{3(2 + \cos x)^6} \quad (3.15)$$

By (2.17), the Weyl scalars of the metric (3.8) read

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \quad \Psi_2 = -\frac{2\Lambda(2 + \cos x)^3 - 3m}{18(2 + \cos x)^3}. \quad (3.16)$$

We now investigate the physical behavior of the space-time described by (3.8).

Fixing y and z , we get the induced metric

$$ds^2 = \left(\frac{\Lambda(2 + \cos x)^2}{3} + \frac{m}{2 + \cos x} \right) dt^2 - 2 \sin x dt dx. \quad (3.17)$$

Consider the null curves in the (t, x) -plane, which are defined by

$$\left(\frac{\Lambda(2 + \cos x)^2}{3} + \frac{m}{2 + \cos x} \right) dt^2 - 2 \sin x dt dx = 0. \quad (3.18)$$

(3.18) gives

$$dt = 0, \quad \frac{dt}{dx} = \frac{6 \sin x (2 + \cos x)}{\Lambda(2 + \cos x)^3 + 3m}. \quad (3.19)$$

(3.19) gives the equations of the projection of the null curves and light-cones in the (t, x) -plane.

Next, we study the geometric behavior of the t -slices.

For any fixed $t \in \mathbb{R}$, it follows from (3.8) that the induced metric of the t -slice reads

$$ds^2 = -(2 + \cos x)^2(dy^2 + dz^2). \quad (3.20)$$

(3.20) describes a three-dimensional cylinder which is periodic in the x -direction.

4 Space-time-periodic solutions

This section is devoted to finding space-time-periodic solutions of the equation (1.2) or (1.3).

4.1 Regular periodic space-times

In particular, we take

$$a = (2 + \cos x)(2 + \sin \frac{\Lambda t}{6}). \quad (4.1)$$

The metric (2.8) becomes

$$ds^2 = \left\{ \frac{\Lambda(2 + \cos x) \cos \frac{\Lambda t}{6}}{3} + \frac{\Lambda(2 + \cos x)^2(2 + \sin \frac{\Lambda t}{6})^2}{3} + \frac{m}{(2 + \cos x)(2 + \sin \frac{\Lambda t}{6})} \right\} dt^2 \\ - 2 \sin x \left(2 + \sin \frac{\Lambda t}{6} \right) dt dx - (2 + \cos x)^2 \left(2 + \sin \frac{\Lambda t}{6} \right)^2 (dy^2 + dz^2).$$

Substituting (4.1) into (2.10)-(2.15) gives

$$R_{0101} = - \frac{\sin^2 x (2 + \sin \frac{\Lambda t}{6})^2 \{ \Lambda(2 + \cos x)^3 (2 + \sin \frac{\Lambda t}{6})^3 + 3m \}}{3(2 + \cos x)^3 (2 + \sin \frac{\Lambda t}{6})^3},$$

$$R_{0202} = - \frac{2\Lambda^2(2 + \cos x)^3 (2 + \sin \frac{\Lambda t}{6})^3 \cos \frac{\Lambda t}{6} - 3m\Lambda \cos \frac{\Lambda t}{6}}{18(2 + \sin \frac{\Lambda t}{6})} \\ - \frac{2\Lambda^2(2 + \cos x)^6 (2 + \sin \frac{\Lambda t}{6})^6 + 3\Lambda m(2 + \cos x)^3 (2 + \sin \frac{\Lambda t}{6})^3 - 9m^2}{18(2 + \cos x)^2 (2 + \sin \frac{\Lambda t}{6})^2},$$

$$R_{0212} = \frac{\sin x \{ 2\Lambda(2 + \cos x)^3 (2 + \sin \frac{\Lambda t}{6})^3 - 3m \}}{6(2 + \cos x)},$$

$$R_{0303} = - \frac{2\Lambda^2(2 + \cos x)^3 (2 + \sin \frac{\Lambda t}{6})^3 \cos \frac{\Lambda t}{6} - 3m\Lambda \cos \frac{\Lambda t}{6}}{18(2 + \sin \frac{\Lambda t}{6})} \\ - \frac{2\Lambda^2(2 + \cos x)^6 (2 + \sin \frac{\Lambda t}{6})^6 + 3\Lambda m(2 + \cos x)^3 (2 + \sin \frac{\Lambda t}{6})^3 - 9m^2}{18(2 + \cos x)^2 (2 + \sin \frac{\Lambda t}{6})^2},$$

$$R_{0313} = \frac{\sin x \{ 2\Lambda(2 + \cos x)^3 (2 + \sin \frac{\Lambda t}{6})^3 - 3m \}}{6(2 + \cos x)},$$

$$R_{2323} = \frac{\Lambda(2 + \cos x)^4 (2 + \sin \frac{\Lambda t}{6})^4}{3} + m(2 + \cos x)(2 + \sin \frac{\Lambda t}{6})$$

and the other $R_{\alpha\beta\mu\nu} = 0$. Substituting (4.1) into (2.16) yields

$$\mathbf{R} \triangleq R^{ijkl} R_{ijkl} = \frac{12m^2}{(2 + \cos x)^6 (2 + \sin \frac{\Lambda t}{6})^6} + \frac{8\Lambda^2}{3}. \quad (4.2)$$

On the other hand, by (2.17), the Weyl scalars of the metric under consideration read

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \quad \Psi_2 = \frac{m}{6(2 + \cos x)^3 (2 + \sin \frac{\Lambda t}{6})^3} - \frac{\Lambda}{9}. \quad (4.3)$$

We now investigate the physical behavior of the space-time.

Fixing y and z , we get the induced metric

$$ds^2 = \left\{ \frac{\Lambda(2 + \cos x) \cos \frac{\Lambda t}{6}}{3} + \frac{\Lambda(2 + \cos x)^2(2 + \sin \frac{\Lambda t}{6})^2}{3} + \frac{m}{(2 + \cos x)(2 + \sin \frac{\Lambda t}{6})} \right\} dt^2 - 2 \sin x \left(2 + \sin \frac{\Lambda t}{6} \right) dt dx.$$

Consider the null curves in the (t, x) -plane, which are defined by

$$\left\{ \frac{\Lambda(2 + \cos x) \cos \frac{\Lambda t}{6}}{3} + \frac{\Lambda(2 + \cos x)^2(2 + \sin \frac{\Lambda t}{6})^2}{3} + \frac{m}{(2 + \cos x)(2 + \sin \frac{\Lambda t}{6})} \right\} dt^2 - 2 \left(2 + \sin \frac{\Lambda t}{6} \right) \sin x dt dx = 0. \quad (4.4)$$

It follows from (4.4) that

$$dt = 0 \quad (4.5)$$

and

$$\frac{dt}{dx} = \frac{6(2 + \cos x)(2 + \sin \frac{\Lambda t}{6})^2 \sin x}{\Lambda(2 + \cos x)^2(2 + \sin \frac{\Lambda t}{6}) \cos \frac{\Lambda t}{6} + \Lambda(2 + \cos x)^3(2 + \sin \frac{\Lambda t}{6})^3 + 3m}. \quad (4.6)$$

(4.4) and (4.5) give the equations of the projection of the null curves in the (t, x) -plane.

4.2 Singular periodic space-times

In particular, we take

$$a = \cos x \sin \frac{\Lambda t}{6}. \quad (4.7)$$

The metric (2.8) becomes

$$ds^2 = \frac{\Lambda \left\{ \frac{1}{2} \cos^2 x \sin \frac{\Lambda t}{3} + \cos^3 x \sin^3 \frac{\Lambda t}{6} + \frac{3m}{\Lambda} \right\}}{3 \cos x \sin \frac{\Lambda t}{6}} dt^2 - 2 \sin x \sin \frac{\Lambda t}{6} dt dx - \cos^2 x \sin^2 \frac{\Lambda t}{6} (dy^2 + dz^2).$$

Substituting (4.7) into (2.10)-(2.15) gives

$$R_{0101} = - \frac{\sin^2 x \sin^2 \frac{\Lambda t}{6} (\Lambda \cos^3 x \sin^3 \frac{\Lambda t}{6} + 3m)}{3 \cos^3 x \sin^3 \frac{\Lambda t}{6}},$$

$$R_{0202} = - \frac{2\Lambda^2 \cos^3 x \sin^3 \frac{\Lambda t}{6} \cos \frac{\Lambda t}{6} - 3\Lambda m \cos \frac{\Lambda t}{6} + 2\Lambda^2 \cos^4 x \sin^5 \frac{\Lambda t}{6} + 3\Lambda m \cos x \sin^2 \frac{\Lambda t}{6}}{18 \sin \frac{\Lambda t}{6}} + \frac{m^2}{2 \cos^2 x \sin^2 \frac{\Lambda t}{6}},$$

$$\begin{aligned}
R_{0212} &= \frac{\sin x \sin \frac{\Lambda t}{6} (2\Lambda \cos^3 x \sin^3 \frac{\Lambda t}{6} - 3m)}{6 \cos x \sin \frac{\Lambda t}{6}}, \\
R_{0303} &= -\frac{2\Lambda^2 \cos^3 x \sin^3 \frac{\Lambda t}{6} \cos \frac{\Lambda t}{6} - 3\Lambda m \cos \frac{\Lambda t}{6} + 2\Lambda^2 \cos^4 x \sin^5 \frac{\Lambda t}{6} + 3\Lambda m \cos x \sin^2 \frac{\Lambda t}{6}}{18 \sin \frac{\Lambda t}{6}} \\
&\quad + \frac{m^2}{2 \cos^2 x \sin^2 \frac{\Lambda t}{6}}, \\
R_{0313} &= \frac{\sin x \sin \frac{\Lambda t}{6} (2\Lambda \cos^3 x \sin^3 \frac{\Lambda t}{6} - 3m)}{6 \cos x \sin \frac{\Lambda t}{6}}, \\
R_{2323} &= \frac{\Lambda \cos^4 x \sin^4 \frac{\Lambda t}{6}}{3} + m \cos x \sin \frac{\Lambda t}{6}
\end{aligned}$$

and the other $R_{\alpha\beta\mu\nu} = 0$. Substituting (4.7) into (2.16) yields

$$\mathbf{R} \triangleq R^{ijkl} R_{ijkl} = \frac{8\Lambda^2}{3} + \frac{12m^2}{\cos^6 x \sin^6 \frac{\Lambda t}{6}} \quad (4.8)$$

On the other hand, by (2.17), the Weyl scalars of the metric under consideration read

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \quad \Psi_2 = -\frac{2\Lambda}{9} + \frac{m}{6 \cos^3 x \sin^3 \frac{\Lambda t}{6}}. \quad (4.9)$$

Obviously, the points $\{t = \frac{6k\pi}{\Lambda} \mid k \in \mathbb{Z}\}$ and $\{x = k\pi + \frac{\pi}{2} \mid k \in \mathbb{Z}\}$ are the physical singularities.

We now investigate the physical behavior of the space-time.

In the present situation,

$$g_{00} = \frac{\Lambda \left(\frac{1}{2} \cos^2 x \sin \frac{\Lambda t}{3} + \cos^3 x \sin^3 \frac{\Lambda t}{6} + \frac{3m}{\Lambda} \right)}{3 \cos x \sin \frac{\Lambda t}{6}}. \quad (4.10)$$

Case 1: $m \geq \frac{\Lambda}{2}$

In this case,

$$\frac{3m}{\Lambda} \geq \frac{3}{2}.$$

It is easy to see that the numerator of g_{00} is larger than zero. In order to check if t is the time variable, it suffices to discuss the sign of the denominator of (4.10). The sign of g_{00} is shown in the Figure 1. In the domain with the sign “+”, t stands for the time variable, however, in the domain with the sign “−”, t is not the time variable.

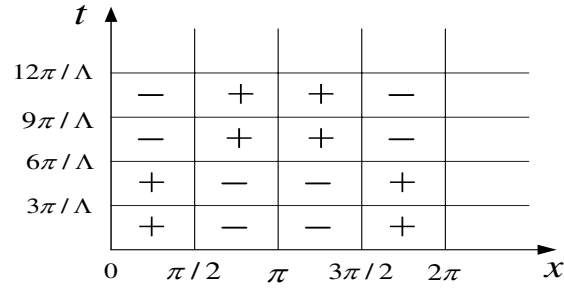


Figure 1: The symbol of g_{00} in the (t, x) -plane for the case $m \geq \frac{\Lambda}{2}$

Case 2: $0 < m < \frac{\Lambda}{2}$

In this case, the symbol of g_{00} is shown in the Figures 2-3.

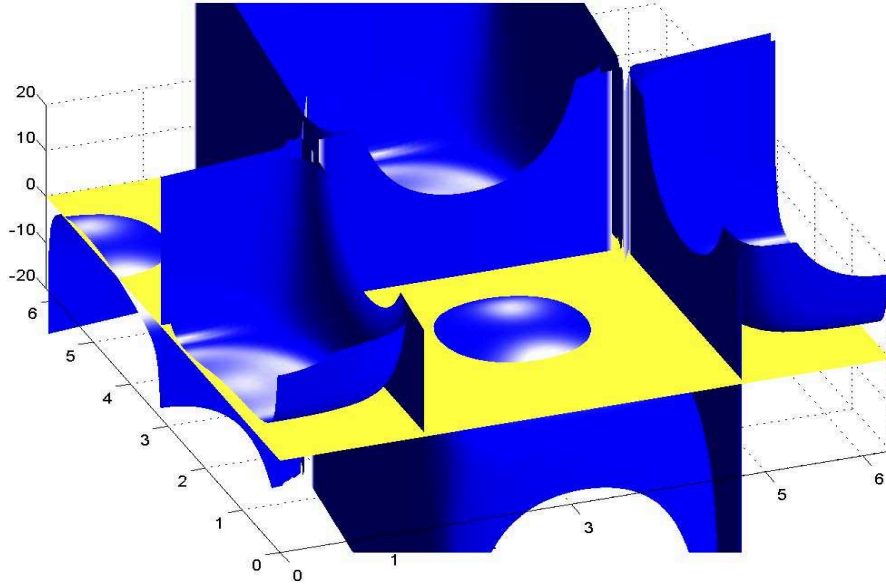


Figure 2: The symbol of g_{00} in the (t, x) -plane for the case $0 < m < \frac{\Lambda}{2}$: one-period

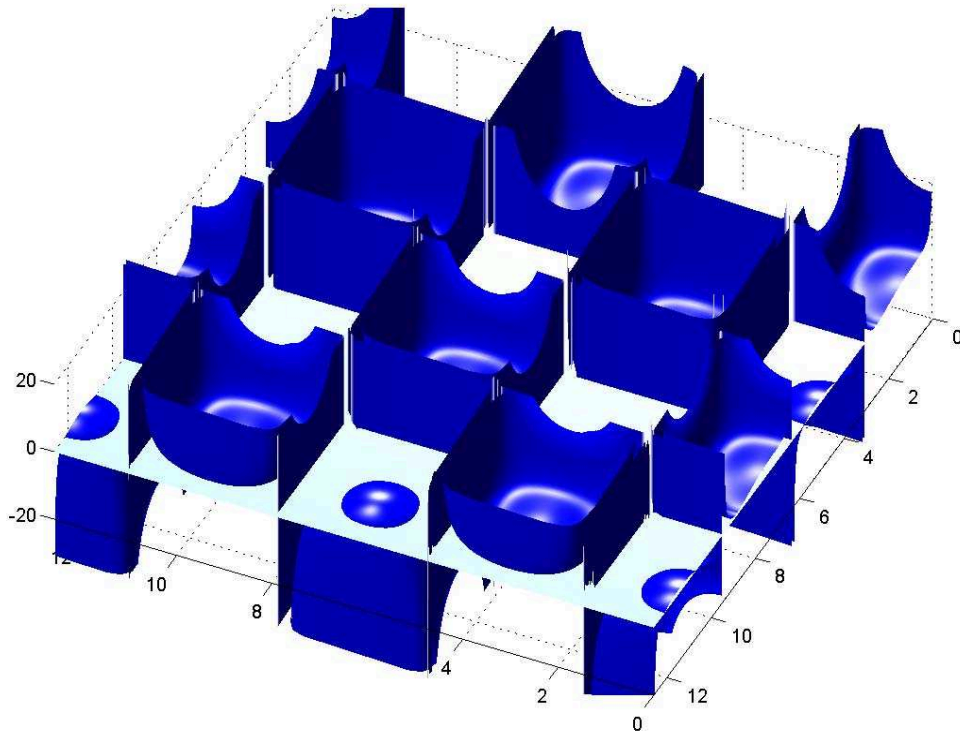


Figure 3: The symbol of g_{00} in the (t, x) -plane for the case $0 < m < \frac{\Lambda}{2}$: multi-period

Remark 4.1 For the above two cases, we can easily prove that, when $g_{00} > 0$, t stands for the time variable; while when $g_{00} < 0$, x stands for the time variable.

Fixing y and z , we get the induced metric

$$ds^2 = \frac{\Lambda \left(\cos^2 x \sin \frac{\Lambda t}{6} \cos \frac{\Lambda t}{6} + \cos^3 x \sin^3 \frac{\Lambda t}{6} + \frac{3m}{\Lambda} \right)}{3 \cos x \sin \frac{\Lambda t}{6}} dt^2 - 2 \sin x \sin \frac{\Lambda t}{6} dt dx. \quad (4.11)$$

Consider the null curves in the (t, x) -plane, which are defined by

$$\frac{\Lambda \left(\cos^2 x \sin \frac{\Lambda t}{6} \cos \frac{\Lambda t}{6} + \cos^3 x \sin^3 \frac{\Lambda t}{6} + \frac{3m}{\Lambda} \right)}{3 \cos x \sin \frac{\Lambda t}{6}} dt^2 - 2 \sin x \sin \frac{\Lambda t}{6} dt dx = 0. \quad (4.12)$$

(4.12) gives

$$dt = 0, \quad \frac{dt}{dx} = \frac{6 \cos x \sin x \sin^2 \frac{\Lambda t}{6}}{\Lambda \left(\cos^2 x \sin \frac{\Lambda t}{6} \cos \frac{\Lambda t}{6} + \cos^3 x \sin \frac{\Lambda t}{6} + \frac{3m}{\Lambda} \right)}. \quad (4.13)$$

Thus, for any fixed $t \in \mathbb{R}$ with $g_{00} > 0$, the induced metric of the *time*-slice reads

$$ds^2 = -\cos^2 x \sin^2 \frac{\Lambda t}{6} (dy^2 + dz^2). \quad (4.14)$$

While for any fixed $x \in \mathbb{R}$ with $g_{00} < 0$, the induced metric of the *time*-slice is

$$ds^2 = \frac{\Lambda \left(\cos^2 x \sin \frac{\Lambda t}{6} \cos \frac{\Lambda t}{6} + \cos^3 x \sin \frac{\Lambda t}{6} + \frac{3m}{\Lambda} \right)}{3 \cos x \sin \frac{\Lambda t}{6}} dt^2 - \cos^2 x \sin^2 \frac{\Lambda t}{6} (dy^2 + dz^2). \quad (4.15)$$

5 Summary and discussion

In this paper we construct a new kind of solutions of the Einstein's field equations with non-vanishing cosmological constant, which possess some interesting physical properties. In particular, we can construct both regular space-time-periodic solution and space-time-periodic solution with physical singularities. The properties enjoyed by these solutions are investigated and some new physical phenomena are discovered.

It is well known that, in cosmology one interesting open problem is *if there exists a "space-time-periodic" solution to the Einstein's field equations*. In the present paper we construct successfully the space-time-periodic solutions of the Einstein's field equations with non-vanishing cosmological constant, this implies the existence of space-time-periodic universe. We learned that the recent observation data show that our Universe has some space-periodic properties (see [9]). Anyway, the space-time-periodic solutions of the Einstein's field equations describe a very interesting kind of physical space-times, and play an important role in modern cosmology and general relativity. We expect some applications of these new phenomena and the space-time-periodic solutions presented in this paper to modern cosmology and general relativity.

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